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Finite Topological Spaces: Coarser and Finer Structures with Connections to Graph Theory

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Abstract **- Finite topological spaces provide a tractable setting for studying fundamental topological properties and their interactions with other mathematical structures. In this paper, we focus exclusively on finite topological spaces, emphasizing the role of coarser and finer topologies, as well as the extremal cases of discrete and indiscrete topologies. Special attention is given to T_0 (Kolmogorov) spaces, which are central in the finite context, since any finite T_1 space is necessarily discrete. We also highlight the significance of the Sierpiński space as the smallest non-discrete and non-indiscrete topology, noting its importance as a classifying space for open sets and its connections to semantics and computational theory. The study further situates finite topological spaces within ongoing research that links topology and graph theory. Various constructions of topologies derived from graph-theoretic concepts are reviewed, including those based on closed neighborhoods, subbases, open hop neighborhoods, and monophonic eccentric neighborhoods. These approaches have led to characterizations of graphs that induce discrete or indiscrete topologies, as well as to the development of compatible topologies where graph connectivity corresponds to topological connectedness. Recent investigations into discrete topological graphs, domination in discrete topological spaces, and special intersection graphs are also discussed. Collectively, these works demonstrate the growing interplay between finite topology and graph theory and motivate further exploration of their combined structures and applications.**

Keywords: **Finite Topological Spaces, T_0 (Kolmogorov) Spaces, Graph-Induced Topologies, Discrete and Indiscrete Topologies, Sierpiński Space**

I. INTRODUCTION

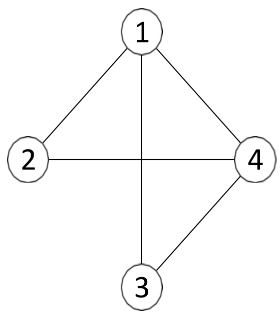
A topological space with a finite underlying point set is called a finite topological space. In this paper, we consider only finite topological spaces. Let X be a finite set and T_1 and T_2 be two topological spaces defined on X; then T_1 is a coarser topology than T_2 if $T_1 \subseteq T_2$, and T_1 is a finer topology than T_2 if $T_2 \subseteq T_1$. The discrete topology is the finest topology that can be given on a set X, and the indiscrete topology is the coarsest topology that can be defined on a point set X. A space X with topology T where there is at least one neighborhood for each pair of distinct points in X is called a T_0 space or Kolmogorov space. A Sierpinski space is the smallest topological space that is neither discrete nor indiscrete. This space has two points, but only one of them is closed. It is the smallest example of a topological space that is neither discrete nor trivial. Because the Sierpiński space is the classifying space for open sets in the Scott topology, it has significant connections to semantics and computational theory. General references and definitions may be found in [1], and an excellent guide to finite topological spaces is given by Stong[2]. When finite topological spaces are concerned, T_0 spaces are of utmost importance. T_1 space is discrete in finite topological spaces. Diesto and Gervacio [3] established topologies using closed neighborhoods as the basis for the topology. Conoy and Lemence used the subbases to do more research.

In 2019, Nianga and Canoy [4] created a topology by utilizing the open hop neighborhood of a vertex set. In 2021, Gamorez and Canoy [5] constructed a topology using monophonic eccentric neighborhoods on a vertex set of the graph and further characterized the graphs that induce discrete and indiscrete topologies. Lemence [6] studied the topologies generated by a few special graphs and also characterizing the graphs that provide topological spaces that are discrete and indiscrete. In 1992, Prea[7] defined a topology compatible with a given graph such that an induced subgraph is connected if and only if it is associated with this topology. In recent years, more attempts have been made to connect topology and graph theory. studies discrete topological graphs and the domination in discrete topological space [8, 9]. Omran *et* al. [10] defined another class of graphs called special intersection graphs in topological graphs.

II. TOPOLOGY INDUCED BY THE GRAPH

Let G be the graph and V be the vertex set of graph G; here we consider the point set $X = V(G)$. Let I_G be collection of topologies on X such that $T \in \mathrm{I}_G$ iff for any x adjacent to y in G then $\exists$ an open set G in X such that $x \in G$ and $y \notin G$ or $x \notin G$ and $y \in G$. We call this collection I_G a section of the Graph induced topology of the graph G. The minimal topology (Minimal in terms of coarser, i.e.,T is minimal if there does not exist any topology $T' \in \mathrm{I}_G$ such that $T' \subset T$.) T is called Graph, the induced topology of the graph G. Let G be any graph of order n. Then clearly, the discrete topology on the point set containing n elements is always a member of I_G and hence I_G is always nonempty. Hence, given a graph G, we always get a topology induced by the graph.

T_0 spaces are foundational when it comes to separation axioms, but T_0 spaces are more significant in finite topological spaces, as T_1 spaces are discrete. Discrete and indiscrete topologies are two extreme topological spaces. Therefore, we define the topology induced by the graph in such a way that the non-trivial graphs of finite order lie well within the two extremes. This gives us a wealth of topics to study. For Example: G be the graph given below.



G

$X = \{1, 2, 3, 4\}$ be the point set and $T = \{\Phi, \{1\}, \{1, 4\}, X\}$ is the topology Induced by the graph.

Proof: Let I_G as a collection of topologies induced by the graph G. Clearly, T is a topology. We need to show T is the minimal topology in I_G. Let T' be any topology strictly coarser than T, i.e., $T' \subset T$. Suppose $T' = \{\Phi, \{1\}, X\}$. Since $\{3, 4\} \in E(G)$ but there does not exists any open set $O \in T'$ such that $3 \in O$ and $4 \notin T' \notin I_G$. O or $3 \notin O$ and $4 \in O$. Hence exists anyopen set $O \in T'$ such that $3 \in O$ and $4 \notin O$ or $3 \notin O$ and $4 \in O$. Hence $T' \notin I_G$.

If suppose $T' = \{\Phi, \{1, 4\}, X\}$. Since $\{1, 4\} \in E(G)$ but there does not exists any open set $O \in T'$ such that $1 \in O$ and $4 \notin O$ or $1 \notin O$ and $4 \in O$. Hence $T' \notin I_G$. If suppose $T' = \{\Phi, X\}$. Since $\{1, 4\} \in E(G)$ but there does not exists any open set $O \in T'$ such that $1 \in O$ and $4 \notin O$ or $1 \notin O$ and $4 \in O$. Hence $T' \notin I_G$. Hence T is the minimal topology which is in I_G and hence T is the topology induced by the graph G.

III. TOPOLOGY INDUCED BY SOME STANDARD CLASS OF GRAPHS

A. Theorem 3.1.

$X = \{1, 2, .., n\}$ be the point set and $T = \{\Phi, X\}$ be the indiscrete topology induced by the graph G. Then G must be a null graph.

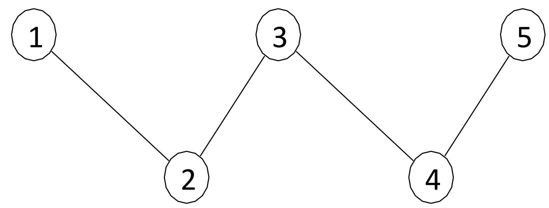
Proof: Let us assume that G is not a null graph. Then $\exists$ an edge $e = xy$ in G, since T is the topology induced by the graph G $\exists$ an open set Q in T $\ni$: $x \in Q$ and $y \notin Q$ or $x \notin Q$ and $y \in Q$. Hence $Q \neq \phi$ and $Q \neq X$, and Q is an open set in X, which contradicts the fact that T is an indiscrete topology. Hence, G must be a null graph.

B. Theorem 3.2.

Let G with vertex set $V(G) = \{1, 2, .., n\}$ be a null graph then I_G consists of all topology with point set $X = V(G)$.

Proof: Let $X = \{1, 2, .., n\}$ be the point set. T be any topology defined on X since G has no edges $T \in I_G$ and indiscrete topology is contained in every topology. Hence, the indiscrete topology will be the topology induced by the null graph.

C. Theorem 3.3.



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Let $G = K_n$ represent the complete graph of order n, then the collection of graph induced topologies I_G consists of only T_0 spaces. Moreover, the topology induced by the graph G is a T_0 or Kolmogorov space.

Proof: Let $T \in I_G$ be arbitrary, since G is complete graph for any $p \neq q$, p is adjacent to q hence $\exists$ an open set $Q \in T$ $\ni$: $i \in Q$ and $j \notin Q$ or $i \notin Q$ and $j \in Q$. Thus T is a T_0 space. Given a graph G, any two topologies induced by the graph G may not be homeomorphic. For example: Consider the path graph G with 5 vertices. Then $T_1 = \{\phi, \{1, 3, 5\}, X\}$ and $T_2 = \{\phi, \{2, 4\}, X\}$ are topology induced by the graph G which are not homeomorphic to each other.

D. Theorem 3.4.

$G = K_n$ be a complete graph of order n then for any $T \in I_G$ $\exists$ an open set with a single element.

Proof: $V(G) = X = \{1, 2, .., n\}$ is the point set and G is complete graph. $T \in I_G$ be arbitrary, Let $Q_1 \in T$ if Q_1 is singleton set then we are done if not say $i, j \in Q_1$, since G is complete i is adjacent to j and hence $\exists$ an open set Q' $\ni$: $i \in Q'$ and $j \notin Q'$ or $i \notin Q'$ and $j \in Q'$. Consider $Q_2 = Q_1 \cap Q'$. Clearly, Q_2 is a non-empty proper subset of Q_1. If Q_2 is singleton then we are done if not then $\exists$ $k, l \in Q_2$ $\ni$: k is adjacent to l hence $\exists$ an open set $\in T$ $\ni$: $k \in Q'$ and $l \notin$ $'$ or $k \notin$ and $l \in Q'$ Now consider $Q_3 = Q_2 \cap Q'$ clearly Q_3 is a non empty proper subset of Q_2. If Q_3 is a singleton set, then we are done; if not, continue the process. Since T is a finite topological space, the process must terminate after a finite number of times, resulting in a singleton set.

Corollary 3.4.1.

If T is a finite T_0 space, then $\exists$ an open set with a *single element. Proof.* Proof follows from Theorems 3.3 and 3.4

E. Theorem 3.5.

G be the graph and T be a topology induced by the graph G. Then T^c defined by $T^c = \{G \subseteq X \mid G = O^c$ where $O \in T\}$ is also a topology induced by the same graph G.

Proof: Firstly T^c is also a topology since $\Phi, X \in T \Longrightarrow \Phi^c = X, X^c = \Phi \in T^c$. T^c is the collection of closed sets of T. Hence, finite union and finite intersection of closed sets are closed, and T^c is a topology on X. Next we claim the following (i) $T^c \in I_G$ (ii) T^c is minimal. Proof of (i) For any edge $\{x, y\} \in E(G)$ since T is a topology induced by the graph G $\exists$ an open set say $O \in T \ni: x \in O$ and $y \notin O$ or $x \notin O$ and $y \in O$. If $x \in O$ and $y \notin O$ then $x \notin O^c$ and $y \in O^c$ hence $T^c \in I_G$. If $x \notin O$ and $y \in O$ then $x \in O^c$ and $y \notin O^c$ hence $T^c \in I_G$.

Proof of (ii) Assume T^c is not minimal then $\exists$ a topology $T_1 \in I_G \ni: T_1 \subset T^c$ then
$T^c \subset (T^c)^c \Longrightarrow T^c \subset T$

$T^c \in I_G$ by claim (i) which contradicts the fact that T is the minimal topology. Hence, T^c 1, 1 is also a topology induced by the graph G.

Corollary 3.5.1.

If T is a T_0 space, then T^c is also a T_0 space. Proof. Proof follows from Theorems 3.3 and 3.5

F. Theorem 3.6.

If G is a bipartite graph that has bipartitions A and B, then $T = \{\Phi, A, X\}$ is the topology induced by the graph G.

Proof: Clearly T is a topology, For any edge say $\{x, y\} \in E(G)$ since G is bipartite say $x \in A$ and $y \in B$, if $x \in A$ then $y \notin A$, hence $T \in I_G$ and T is minimal, if not then $\exists$ a topology $T' \subset T \ni: T' \in I_G$ then $T' = \{\Phi, X\}$ hence for any edge $\{x, y\} \in E(G)$ there does not exists any open set O in T such that $x \in O$ and $y \notin O$ or $x \notin O$ and $y \in O$ this leads a contradiction. Hence, T is minimal and T is the topology induced by the graph G is bipartite.

Corollary 3.6.1.

If G is bipartite with bipartition A and B then $T_1 = \{\Phi, A, X\}$ and $T_2 = \{\phi, B, X\}$ are topology induced by the graph G where $X = A \cup B$. Proof. The proof follows from Theorem 3.5 and Theorem 3.6

G. Theorem 3.7.

Let $G = C_{2n+1}$ be a cyclic graph with edges connecting $1{-}2{-}3 \ldots 2n{-}2n{+}1{-}1$ *then $T = \{\Phi, \{1\}, A, X\}$ where $A = \{1, 3, 5, \ldots, 2n+1\}$.*

Proof: Clearly, T is a topology. For any edge $e = \{i, j\} \in E\ (C_{2n+1})$ other than $\{1, 2n+1\}$, one vertex is odd and another is even; therefore, the odd vertex $\in A$ and the even vertex do not. For the edge $\{1, 2n+1\}$1 $\in \{1\}$ and $2n+1 \notin \{1\}$ hence $T \in I_G$. Next, we claim that T is minimal. If not $\exists$ a topology $T' \in I_G$ such that $T' \subset T$ then the possibility for T' is $T' = \{\Phi, X\}$ or $T' = \{\Phi, \{1\}, X\}$ or $T' = \{\Phi, A, X\}$. If $T' = \{\Phi, X\}$ then for any edge $e = \{x, y\} \in E(G)$ then there does not exists any open set O in T' such that $x \in O$ and $y \notin O$.If $T' = \{\Phi, \{1\}, X\}$ then for edge $e = \{2, 3\} \in E(G)$ then there does not exists any open set O in T' such that $2 \in O$ and $3 \notin O$. If $T' = \{\Phi, A, X\}$ then for edge $e = \{1, 2n+1\} \in E(G)$ then there does not exist any open set O in T' such that $1 \in O$ and $2n+1 \notin O$. Hence T is the minimal topology in I_G and T is the topology induced by the graph G.

H. Theorem 3.8.

G be a tripartite graph with tripartition A, BandC then the topology induced by the graph G is $T = \{\phi, A, A \cup B, X\}$

Proof: Clearly, T is a topology. Since T is a tripartite graph, we have 3 kinds of edges. first kind whose end vertices are in A and B, second kind whose end vertices are in B and C, third kind whose end vertices are in A and C. For any edge $e = \{x, y\}$ in the first kind, say $x \in A$ and $y \in B$ Hence $\exists$ an open set $A \in T \ni: x \in A$ and $y \notin A$. If $e = \{x, y\}$ is an edge of the second kind, then $x \in B$ and $y \in C$ then we take $Q = A \cup B$ such that $x \in Q$ and $y \notin Q$. If $e = \{x, y\}$ is an edge of the third kind, then $x \in Q$ and $yinC$ then we take $Q = A$ such that $x \in Q$ and $y \notin Q$. Hence $T \in I_G$. Now we will show T is minimal, Assume that T is not minimal then $\exists$ $T' \in I_G$ such that T' is proper subset of T then possible T' are $T' = \{\Phi, X\}$ or $T' = \{\Phi, A, X\}$ or $T' = \{\Phi, A \cup B, X\}$ If $T' = \{\Phi, X\}$ then for any edge $e = \{x, y\} \in E(G)$ then there does not exists any open set Q in T' such that $x \in Q$ and $y \notin Q$. If $T' = \{\Phi, A, X\}$ then for any edge $e = \{x, y\} \in E(G)$ of second kind then $x \in B$ and $y \in C$ then there does not exists any open set Q in T' such that $x \in Q$ and $y \notin Q$. If $T' = \{\Phi, A \cup B, X\}$ then for any edge $e = \{x, y\} \in E(G)$ of first kind then $x \in B$ and $y \in C$ then there does not exists any open set Q in T' such that $x \in Q$ and $y \notin Q$. Hence T is the minimal topology in I_G and T is the topology induced by the graph G.

I. Theorem 3.9.

Let Let G be a graph and $T_G = \{\Phi, X, A_1, A_2, \ldots, A_k\}$ be the graph induced topology of graph G on point set $X = V(G)$ then $T = T_G \cup \{Y\}$ is the graph induced topology of graph $G + \{v\}$ where $Y = X \cup \{v\}$.

Proof: Here we claim the following. (i) T is topology on Y. (ii) $T \in I_{G+\{v\}}$. (iii) T is minimal topology. Proof of claim (i) Φ and Y are in T. Let G and H be any two open sets in T

case 1: If $Q_1 = \Phi$ and Q_2 be any open set then $Q_1 \cup Q_2 = Q_2$ and $Q_1 \cap Q_2 = \Phi$ both are trivially in T.
case 2: If $Q_1 = A_i$ and $Q_2 = A_j$ then $Q_1 \cup Q_2$ and $Q_1 \cap Q_2$ are in T_G since T_G is a topology and hence $Q_1 \cap Q_2$ are in T.
case 3: If $Q_1 = X$ and $Q_2 = A_i$ then $Q_1 \cup Q_2 = X$ and $Q_1 \cap Q_2 = Q_2$ and hence in T. case 4: If $Q_1 = Y$ and $Q_2 = A_i$ then $Q_1 \cup Q_2 = Y$ and $Q_1 \cap Q_2 = Q_2$ and hence in T. case 5: If $Q_1 = X$ and $Q_2 = Y$ then $Q_1 \cup Q_2 = Y$ and $Q_1 \cap Q_2 = X$ and hence in T. Which proves claim (i). Proof of claim (ii) Let $e = \{x, y\} \in E(G + \{v\})$ if e is an edge in G and T_G is the topology induced by the graph G $\exists$ an open set $O \in T$ such that $x \in O$ and $y \notin O$ or $x \notin O$ and $y \in O$ and $O \in T$ we are done.

If e is an edge with one vertex v and another vertex, say x in $V(G)$, then choose an open set as X $x \in X$ and $v \notin X$. hence for any edge in $E(G + \{v\})$ we can find an open set O with one vertex in O and other not in O hence proof the claim (ii). proof of claim (iii) Suppose assume that T is not minimal then $\exists$ a topology $T' \in I_{G+\{v\}}$ such that $T' \subset T$. Consider $A = \{A_1, A_2, ..., A_k\}$. Suppose $T' = \{\Phi, Y, X\}$ then for any edge $e = \{x, y\} \in E(G + \{v\})$ then there does not exists any open set O in T' such that $x \in O$ and $y \notin O$ or $x \notin O$ and $y \in O$ thus leading a contradiction.

Suppose $T' = T - \{X\}$ then clearly T' is a topology and for any $i \in V(G)$, v is adjacent to i in G+{v} then there exsits an open set $O \in T'$ such $i \in O$ and $v \notin O$ or $i \notin O$ and $v \in O$ for all $i \in V(G)$ since Y is the only open set which contain v, Y also contain all vertices of $V(G)$. Hence $Y \not\models O$ and O must be equal to some A_i and $i \in A_i$ for each $i \in V(G)$. And $X = \cup_{i \in V(G)} A_i$ hence $X \in T'$ leading to contradiction. Let $T' = \{\Phi, X\} \cup A'$ where $A' \subset A$ now consider $T' = \{\Phi, X\} \cup A'$ then T' is a topology and for any edge $e = \{x, y\} \in E(G) \implies e\{x, y\} \in E(G + \{v\})$ also and T' is topology induced by the graph $G + \{v\}$ hence $\exists$ open set $O \in T'$ such that $x \in O$ and $y \notin O$ or $x \notin O$ and $y \in O$ and hence $O \in T'$ and hence $T' \in I_G$ which contradicts the minimality of T_G. Hence the proof.

Corollary 3.9.1.

Let G be the graph and $T = \{\Phi, X, A_1, A_2, ..., A_k\}$ be the topology induced by the graph G then the topology induced by the graph $G + K_n$ will be given by $T_n = \{\Phi, X \cup \{v_1\}, X \cup \{v_1, v_2\}, ..., X \cup \{v_1, v_2, ..., v_n\}, X, A_1, A_2, ..., A_k\}$ where $V(K_n) = \{v_1, v_2, ..., v_n\}$.

Proof: By theorem 3.9 if $T = \{\Phi, X, A_1, A_2, ..., A_k\}$ is the topology induced by the graph G then $T_1 = \{\Phi, X \cup \{v_1\}, X, A_1, A_2, ..., A_k\}$ is the topology induced by the graph $G + K_1$ where $V(K_1) = \{v_1\}$ by induction we can say that the topology induced by the graph $(G + K_1) + K_1$ is $T_2 = \{\Phi, X \cup \{v_1\}, X \cup \{v_1, v_2\}, X, A_1, A_2, ..., A_k\}$ where $V(K_n) = \{v_1, v_2, ..., v_n\}$ and $(G + K_1) + K_1 = (G + K_2)$ then continue the process after n steps we get that the topology induced by the graph $G + K_n$ is given by $T_n = \{\Phi, X \cup \{v_1\}, X \cup \{v_1, v_2\}, ..., X \cup \{v_1, v_2, ..., v_n\}, X, A_1, A_2, ..., A_k\}$ where $V(K_n) = \{v_1, v_2, ..., v_n\}$.

IV. CONCLUSION

In this paper, we introduced and studied a notion of topology induced by a graph on a finite vertex set, emphasizing the role of minimal topologies that respect adjacency relations in the underlying graph. We characterized the induced topologies for several standard classes of graphs, including null graphs, complete graphs, bipartite graphs, odd cycles, tripartite graphs, and graph extensions via joins. Overall, this work highlights a clear and constructive bridge between finite topology and graph theory. The framework developed here not only unifies several existing constructions of graph-based topologies but also opens avenues for further research.

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Use of Artificial Intelligence (AI)-Assisted Technology for Manuscript Preparation
The authors confirm that no AI-assisted technologies were used in the preparation or writing of the manuscript, and no images were altered using AI.

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